Probability and Measure 2021-2022
Exam 1

1. **[30 points]** Let A be a set in the Borel σ-algebra of $\mathbb{R}$.

   (a) Prove that the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = m(A \cap (-\infty, x])$ is continuous (m denotes Lebesgue measure).

   (b) Assume that A has positive Lebesgue measure. Prove that for every $\delta > 0$, there exists $x \in \mathbb{R}$ such that $m(A \cap (x, x + \delta)) > 0$.

2. **[30 points]** Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function (with respect to the measure space $(\mathbb{R}^d, \mathcal{B}^d, m)$). Let $E_1, E_2, \ldots$ be Borel subsets of $\mathbb{R}^d$ with the property that every $x \in \mathbb{R}^d$ belongs to at most finitely many of these sets. Prove that

$$\lim_{n \to \infty} \int_{E_n} f \, dm = 0.$$

3. **[20 points]** Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Assume that Ω_2 has at least two elements and $\mathcal{F}_2$ is the trivial σ-algebra, that is, $\mathcal{F}_2 = \{\Omega_2, \varnothing\}$. Prove that if $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{B})$-measurable, then $f(\omega_1, \omega_2)$ does not depend on ω_2 (that is, $f(\omega_1, \omega_2) = f(\omega_1, \omega_2')$ for all ω_1, ω_2, ω_2').

4. **[10 points]** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Give the definition of the set $\mathcal{L}^\infty(\Omega, \mathcal{A}, \mu)$, of the semi-norm $\| \cdot \|_\infty$ and of the set $L^\infty(\Omega, \mathcal{A}, \mu)$.

+10 free points